

Curvaton with Polynomial Potential

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ABSTRACT: In general a weakly self-interacting curvaton field is expected and the curvaton potential takes the polynomial form. The curvaton potential can be dominated by the self-interaction term during the period of inflation if the curvaton field stays at a large vacuum expectation value. We use the $\delta\mathcal{N}$ formalism to calculate the primordial curvature perturbation in the various possible scenarios which make the curvaton model much richer.

KEYWORDS: curvaton, curvature perturbation.

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1. Introduction

Inflation [1, 2, 3] is the leading paradigm to solve the puzzles in the hot big bang model. The quasi-exponential expansion during inflation makes our universe almost homogeneous and isotropic. A bonus of inflation is that the quantum fluctuations of the scalar fields during inflation can naturally explain the small temperature fluctuations in the cosmic microwave background radiations and seed the formation of the large-scale structure. At the leading order, these quantum fluctuations are characterized by their amplitudes and tilts. If we only focus on the primordial power spectrum, we cannot distinguish inflation from curvaton model [4, 5, 6, 7].

The gravitational dynamics itself introduces important non-linearities, which will contribute to the final non-Gaussianity in the large-scale CMB anisotropies. In fact, the CMB non-Gaussianity [8, 9, 10] opens a windows to probe the physics of the early universe. A well-understood ansatz of non-Gaussianity has a local shape. This kind of non-Gaussianity can be characterized by some non-linearity parameters f_{NL} , g_{NL} and so on:

$$\zeta(\mathbf{x}) = \zeta_g(\mathbf{x}) + \frac{3}{5}f_{NL}\zeta_g^2(\mathbf{x}) + \frac{9}{25}g_{NL}\zeta_g^3(\mathbf{x}) + \dots, \quad (1.1)$$

where ζ_g is the linear, Gaussian part of curvature perturbation. The current bound from WMAP 5yr data [11] is $-9 < f_{NL}^{local} < 111$ at 95% CL. Even though a Gaussian distribution is still consistent with the present experiments, much of the allowed region for $f_{NL}^{local} < 0$ from WMAP 3yr data was cut. In the single field inflation model $f_{NL}^{local} \sim \mathcal{O}(n_s - 1)$ [12], which is constrained by WMAP ($n_s = 0.960^{+0.014}_{-0.013}$) [11] to be much less than unity. However the curvaton model can easily generate a large local-type non-Gaussianity [4, 5, 6, 7, 13, 14, 15, 16]. See [17] for a nice review and see [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29] for the recent relevant discussions.

In general we can expect that the curvaton field does not have only a mass term in its potential. Instead of the simplest curvaton potential $\frac{1}{2}m^2\sigma^2$, we adopt a form for the potential which allows a range of possibilities

$$V(\sigma) = \frac{1}{2}m^2\sigma^2 + \sum_{n \geq 4} \lambda_n \frac{\sigma^n}{M^{n-4}}. \quad (1.2)$$

The term with $n > 4$ is non-renormalizable and suppressed by a UV scale M . If all of the interaction terms are negligible, $\frac{\delta\rho_\sigma}{\rho_\sigma} \sim 2\frac{\delta\sigma}{\sigma} + (\frac{\delta\sigma}{\sigma})^2$ and thus the second or higher order non-Gaussianity parameters (g_{NL}, \dots) will be 0. In the literatures the potential of curvaton is assumed to be dominated by the mass term. If a subdominant interaction term is taken into account, the non-linear evolution on large scales is possible. The curvaton dynamics after inflation was discussed in [28, 29, 30, 31, 32]. In this case f_{NL} can be small even when $f_D \ll 1$, but g_{NL} should be large [16, 28, 29]. In all of these papers, the authors only focused on the case where the curvaton potential is always dominated by the mass term and the interaction term is taken as a perturbation. However, the self-interaction term can be dominant if the curvaton mass is small enough and the vacuum expectation value of curvaton during inflation is large enough. If so, the higher order non-Gaussianity parameters are also expected to be larger.

In this paper we will use $\delta\mathcal{N}$ formalism [33, 34, 35] to calculate the primordial curvature perturbation for the curvaton model with a polynomial potential. When the self-interaction term is taken into account, the curvaton model becomes much richer. Our paper is organized as follows. In Sec. 2, we calculate the primordial power spectrum and the non-linearity parameters in various possible scenarios. In Sec. 3, the spectral index of the primordial power spectrum and the enhancement of the second order non-Gaussianity parameters are discussed. The evolution of curvaton before it starts to oscillate, and after it starts to oscillate, but before it decays, are investigated in Sec. 4 and Sec. 5 respectively. In Sec. 6, we give some discussions on curvaton model.

2. Primordial curvature perturbation

In this paper we expand any field or perturbation at each order (n) as follows

$$\zeta(t, \mathbf{x}) = \zeta^{(1)}(t, \mathbf{x}) + \sum_{n=2}^{\infty} \frac{1}{n!} \zeta^{(n)}(t, \mathbf{x}). \quad (2.1)$$

We assume that the first-order term $\zeta^{(1)}$ is Gaussian and higher-order terms describe the non-Gaussianity of the full nonlinear ζ . Working in the framework of Fourier transformation of ζ , the primordial power spectrum \mathcal{P}_ζ is defined by

$$\langle \zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2) \rangle = (2\pi)^3 \mathcal{P}_\zeta(k_1) \delta^3(\mathbf{k}_1 + \mathbf{k}_2), \quad (2.2)$$

and the primordial bispectrum and trispectrum are defined by

$$\langle \zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3) \rangle = (2\pi)^3 B_\zeta(k_1, k_2, k_3) \delta^3(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3), \quad (2.3)$$

$$\langle \zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\zeta(\mathbf{k}_4) \rangle = (2\pi)^3 T_\zeta(k_1, k_2, k_3, k_4) \delta^3(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4). \quad (2.4)$$

The bispectrum and trispectrum are respectively related to the power spectrum by

$$B_\zeta(k_1, k_2, k_3) = \frac{6}{5} f_{NL} [\mathcal{P}_\zeta(k_1)\mathcal{P}_\zeta(k_2) + 2 \text{ perms}], \quad (2.5)$$

$$\begin{aligned} T_\zeta(k_1, k_2, k_3, k_4) &= \tau_{NL} [\mathcal{P}_\zeta(k_{13})\mathcal{P}_\zeta(k_3)\mathcal{P}_\zeta(k_4) + 11 \text{ perms}] \\ &+ \frac{54}{25} g_{NL} [\mathcal{P}_\zeta(k_2)\mathcal{P}_\zeta(k_3)\mathcal{P}_\zeta(k_4) + 3 \text{ perms}]. \end{aligned} \quad (2.6)$$

Here the non-linearity parameter τ_{NL} is not an independent non-linearity parameter and it is given by

$$\tau_{NL} = \frac{36}{25} f_{NL}^2. \quad (2.7)$$

But g_{NL} is an independent parameter which will be calculated in this paper.

The primordial density perturbation can be described in terms of the nonlinear curvature perturbation on uniform density hypersurfaces [36]

$$\zeta(t, \mathbf{x}) = \delta \mathcal{N}(t, \mathbf{x}) + \frac{1}{3} \int_{\bar{\rho}(t)}^{\rho(t, \mathbf{x})} \frac{d\tilde{\rho}}{\tilde{\rho} + \tilde{p}}, \quad (2.8)$$

where $\mathcal{N} = \int H dt$ is the integrated local expansion, $\bar{\rho}$ is the homogeneous density in the background model, $\tilde{\rho}$ is the local density and \tilde{p} is the local pressure.

For simplicity, the potential of curvaton field is assumed to contain a mass term and a self-interaction term as follows

$$V(\sigma) = \frac{1}{2} m^2 \sigma^2 + \frac{1}{n(n-1)} \lambda \sigma^n. \quad (2.9)$$

The coupling constant λ takes dimensions of E^{4-n} . The effective mass of curvaton is given by

$$\tilde{m} = \sqrt{m^2 + \lambda \sigma^{n-2}}. \quad (2.10)$$

The potential is dominated by the interaction term if

$$\sigma > \sigma_c = \left(\frac{n(n-1)m^2}{2\lambda} \right)^{\frac{1}{n-2}}. \quad (2.11)$$

For a weakly coupled field, the quantum fluctuations can be well described by a Gaussian random field [37]. Since we are also interested in the case where the self-interaction term

dominates the potential, we want to estimate the effects of the non-linear quantum fluctuations in curvaton field at Hubble exit during inflation. The fluctuation of curvaton is expanded to the third order as follows

$$\sigma_* = \bar{\sigma}_* + \delta\sigma_*^{(1)} + \frac{1}{2}\delta\sigma_*^{(2)} + \frac{1}{6}\delta\sigma_*^{(3)}, \quad (2.12)$$

where $*$ denotes that the quantities are evaluated at the Hubble exit during inflation. The perturbations of a self-interacting scalar field during inflation are discussed in the Appendix A. The second and the third order perturbations are respectively related to $\delta\sigma_*^{(1)}$ by Eqs.(A.7) and (A.8). Here the curvaton potential deviates from the exactly quadratic form and then the non-linear evolution of curvaton field on large scales is expected. Generally the initial amplitude of curvaton oscillations is some function of the field value at the Hubble exit:

$$\sigma_o = \sigma_o(\sigma_*). \quad (2.13)$$

Thus we can expand σ_o around $\bar{\sigma}_o = \sigma_o(\bar{\sigma}_*)$ as follows

$$\sigma_o = \bar{\sigma}_o \left[1 + X + \frac{1}{2}(h_2 + \kappa_2)X^2 + \frac{1}{6}(h_3 + 3h_2\kappa_2 + \kappa_3)X^3 \right], \quad (2.14)$$

where

$$X = \frac{\delta\sigma_o^{(1)}}{\bar{\sigma}_o}, \quad (2.15)$$

$$h_2 = \frac{\bar{\sigma}_o\sigma_o''}{\sigma_o'^2}, \quad \kappa_2 = -(n-2)\frac{N_k}{3H_*^2}\frac{\lambda\bar{\sigma}_o^{n-2}}{\sigma_o'}, \quad (2.16)$$

$$h_3 = \frac{\bar{\sigma}_o^2\sigma_o'''}{\sigma_o'^3}, \quad \kappa_3 = -(n-2)(n-3)\frac{N_k}{3H_*^2}\frac{\lambda\bar{\sigma}_o^{n-2}}{\sigma_o'^2}. \quad (2.17)$$

Here the prime denotes the derivative with respect to σ_* . Usually $\sigma_o' \sim \mathcal{O}(1)$, and then $-\kappa_2 \sim -\kappa_3 \sim N_k\lambda\bar{\sigma}_o^{n-2}/H_*^2 \lesssim N_k\tilde{m}^2/H_*^2 \ll 1$. So it is also reasonable to consider that the quantum fluctuations of curvaton at Hubble exit during inflation can be well described by a Gaussian random field, namely $\sigma_* = \bar{\sigma}_* + \delta\sigma_*$. The higher order terms can be neglected even when the curvaton self-interaction term dominates its potential. In the following discussions, we will ignore all of the terms with κ_2 and κ_3 . These terms can be recovered by $h_2 \rightarrow h_2 + \kappa_2$ and $h_3 \rightarrow h_3 + 3h_2\kappa_2 + \kappa_3$ if one want.

In order to make our calculations clearer, we calculate the curvature perturbation in the various possible scenarios separately.

2.1 Curvaton potential is dominated by the mass term during inflation

In this case, the value of curvaton during inflation satisfies $\sigma_* \ll \sigma_c$ and the interaction term can be taken as a perturbation. The curvature perturbation in this case has been discussed very well. To make our paper complete, we directly quote the results from [16].

The amplitude of the primordial power spectrum and the non-linearity parameters are respectively given by

$$P_\zeta = \frac{1}{9\pi^2} f_D^2 q^2 \frac{H_*^2}{\sigma_*^2}, \quad (2.18)$$

$$f_{NL} = \frac{5}{4f_D}(1+h_2) - \frac{5}{3} - \frac{5f_D}{6}, \quad (2.19)$$

$$g_{NL} = \frac{25}{54} \left[\frac{9}{4f_D^2}(h_3 + 3h_2) - \frac{9}{f_D}(1+h_2) + \frac{1}{2}(1-9h_2) + 10f_D + 3f_D^2 \right], \quad (2.20)$$

where

$$q = \frac{\sigma_* \sigma'_o}{\sigma_o}, \quad f_D = \frac{3\Omega_{\sigma,D}}{4 - \Omega_{\sigma,D}}, \quad (2.21)$$

where $\Omega_{\sigma,D}$ is the fraction of curvaton energy density in the energy budget at the time of curvaton decay. If the curvaton potential is purely quadratic, $h_2 = h_3 = 0$ and then

$$g_{NL} + \frac{10}{3}f_{NL} \simeq 0. \quad (2.22)$$

Any deviation from the above relation implies that the curvaton potential does not take the purely quadratic form.

2.2 Curvaton potential is dominated by the interaction term during inflation

In this subsection, we focus on the cases in which the self-interaction term dominates the curvaton potential during inflation. The value of curvaton is roughly the same as that when it starts to oscillate. So we also assume that the curvaton energy density is dominated by the self-interaction term when it starts to oscillate. According to Eq.(2.14), the curvaton density fluctuation during the curvaton oscillation can be expanded as

$$\rho_{\sigma_o} = \bar{\rho}_{\sigma_o} \left(1 + nX + \frac{n}{2}(n-1+h_2)X^2 + \frac{n}{6}[(n-1)(n-2) + 3(n-1)h_2 + h_3]X^3 \right). \quad (2.23)$$

As demonstrated in [38], the energy density of an oscillating scalar field in an expanding universe with potential $V \sim \sigma^n$ scales as

$$\rho_\sigma \sim a^{-6n/(n+2)}. \quad (2.24)$$

Or equivalently, the pressure of curvaton when it is oscillating is related to its energy density by

$$p = \frac{n-2}{n+2}\rho. \quad (2.25)$$

When curvaton starts to oscillate, but before it decays, the non-linear curvature perturbation on uniform-curvaton density hypersurfaces is given by

$$\zeta_{\sigma_o}(t, \mathbf{x}) = \delta\mathcal{N}(t, \mathbf{x}) + \frac{n+2}{6n} \int_{\bar{\rho}_{\sigma_o}}^{\rho_{\sigma_o}(t, \mathbf{x})} \frac{d\tilde{\rho}_{\sigma_o}}{\tilde{\rho}_{\sigma_o}}. \quad (2.26)$$

Therefore the curvaton density on spatially flat hypersurfaces is

$$\rho_{\sigma_o}|_{\delta\mathcal{N}=0} = \exp\left[\frac{6n}{n+2}\zeta_{\sigma_o}\right] \bar{\rho}_{\sigma_o}. \quad (2.27)$$

Considering Eq.(2.23), order by order, we obtain

$$\zeta_{\sigma_o}^{(1)} = \frac{n+2}{6}X, \quad (2.28)$$

$$\zeta_{\sigma_o}^{(2)} = -\frac{6}{n+2}(1-h_2)\left(\zeta_{\sigma_o}^{(1)}\right)^2, \quad (2.29)$$

$$\zeta_{\sigma_o}^{(3)} = \left(\frac{6}{n+2}\right)^2 (2-3h_2+h_3)\left(\zeta_{\sigma_o}^{(1)}\right)^3. \quad (2.30)$$

The energy density of the oscillating curvaton decreases as $\rho_\sigma \sim a^{-6n/(n+2)}$. If $n < 4$, the energy density of curvaton increases with respect to radiation, but decreases with respect to radiation if $n > 4$. On the other hand, the amplitude of the curvaton oscillations also decreases, and it is possible that the self-interaction term becomes subdominant before it decays. But it is also possible that this transition does not happen before curvaton decays. We will investigate these two possibilities in Sec. 5 in detail. Here we calculate the primordial curvature perturbation for these two cases respectively.

2.2.1 Curvaton potential is dominated by the interaction term before it decays

The curvaton-decay hypersurface is a uniform-density hypersurface and thus from Eq.(2.8) the perturbed expansion on this hypersurface is $\delta\mathcal{N} = \zeta$, where ζ is the total curvature perturbation at curvaton-decay hypersurface. Before the curvaton decays, there have been radiations which are the productions of inflaton decay. Since the equation of state of radiations is $p_r = \rho_r/3$, the curvature perturbation related to radiations is

$$\zeta_r = \zeta + \frac{1}{4} \ln \frac{\rho_r}{\bar{\rho}_r}. \quad (2.31)$$

The pressure of the oscillating curvaton is $p = \frac{n-2}{n+2}\rho$ and thus

$$\zeta_{\sigma_o} = \zeta + \frac{n+2}{6n} \ln \frac{\rho_{\sigma_o}}{\bar{\rho}_{\sigma_o}}. \quad (2.32)$$

In the absence of interactions between radiations and curvaton, the curvature perturbations ζ_r and ζ_{σ_o} are conserved respectively and the above two equations can be written as

$$\rho_r = \bar{\rho}_r \exp[4(\zeta_r - \zeta)], \quad (2.33)$$

$$\rho_{\sigma_o} = \bar{\rho}_{\sigma_o} \exp\left[\frac{6n}{n+2}(\zeta_{\sigma_o} - \zeta)\right]. \quad (2.34)$$

At the time of curvaton decay, the total energy density ρ_{tot} is conserved, i.e.

$$\rho_r(t_D, \mathbf{x}) + \rho_{\sigma_o}(t_D, \mathbf{x}) = \bar{\rho}_{tot}(t_D). \quad (2.35)$$

Requiring that the total energy density is uniform on the decay surface, we have

$$(1 - \Omega_{\sigma,D}) e^{4(\zeta_r - \zeta)} + \Omega_{\sigma,D} e^{\frac{6n}{n+2}(\zeta_{\sigma o} - \zeta)} = 1, \quad (2.36)$$

where $\Omega_{\sigma,D} = \bar{\rho}_{\sigma,D}/\bar{\rho}_{tot}$ is the fraction of curvaton energy density in the energy budget at the time of curvaton decay. Here we assume the curvaton suddenly decays into radiation. In curvaton model, usually we also assume the curvature perturbation generated by inflaton is very small and can be ignored, e.g. $\zeta_r = 0$. Order by order from Eq.(2.36), we have

$$\zeta^{(1)} = f_D \zeta_{\sigma o}^{(1)}, \quad (2.37)$$

$$\zeta^{(2)} = \left[\frac{6(n-1+h_2)}{(n+2)f_D} - \frac{8(n-1)}{n+2} - \frac{2(4-n)}{n+2} f_D \right] \left(\zeta^{(1)} \right)^2, \quad (2.38)$$

$$\begin{aligned} \zeta^{(3)} = & \left[\frac{36}{(n+2)^2 f_D^2} [(n-1)(n-2) + h_3 + 3(n-1)h_2] \right. \\ & - \frac{144}{(n+2)^2 f_D} (n-1)(n-1+h_2) + \frac{4}{(n+2)^2} [44n^2 - 121n + 68 - 9(4-n)h_2] \\ & \left. - \frac{80}{(n+2)^2} (n-1)(n-4)f_D + \frac{12}{(n+2)^2} (4-n)^2 f_D^2 \right] \left(\zeta^{(1)} \right)^3, \end{aligned} \quad (2.39)$$

where

$$f_D = \frac{3n\Omega_{\sigma,D}}{2(n+2) - (4-n)\Omega_{\sigma,D}}. \quad (2.40)$$

Therefore the amplitude of the primordial power spectrum is

$$P_\zeta = \left(\frac{n+2}{12\pi} \right)^2 f_D^2 q^2 \frac{H_*^2}{\sigma_*^2}. \quad (2.41)$$

Identifying $\zeta^{(1)} = \zeta_g$ and recalling $\zeta = \zeta^{(1)} + \frac{1}{2}\zeta^{(2)} + \frac{1}{6}\zeta^{(3)}$, from Eq.(1.1) the non-linearity parameters are given by

$$f_{NL} = \frac{5}{6} \left[\frac{6(n-1+h_2)}{(n+2)f_D} - \frac{8(n-1)}{n+2} - \frac{2(4-n)}{n+2} f_D \right], \quad (2.42)$$

$$\begin{aligned} g_{NL} = & \frac{25}{54} \left[\frac{36}{(n+2)^2 f_D^2} [(n-1)(n-2) + h_3 + 3(n-1)h_2] \right. \\ & - \frac{144}{(n+2)^2 f_D} (n-1)(n-1+h_2) + \frac{4}{(n+2)^2} [44n^2 - 121n + 68 - 9(4-n)h_2] \\ & \left. - \frac{80}{(n+2)^2} (n-1)(n-4)f_D + \frac{12}{(n+2)^2} (4-n)^2 f_D^2 \right]. \end{aligned} \quad (2.43)$$

For $n = 2$, these results are just the same as those in Sec. 2.1. For $n \neq 2$, if $f_D \ll 1$, $g_{NL} \simeq 50(n-1)(n-2)/(3(n+2)^2 f_D^2)$ which is large, and

$$g_{NL} \simeq \frac{2(n-2)}{3(n-1)} f_{NL}^2. \quad (2.44)$$

For $n > 2$, g_{NL} is positive.

These results can be easily understood. The energy density of curvaton is $\rho_\sigma \sim \sigma^n$. Considering $\sigma \rightarrow \sigma + \delta\sigma$, we have

$$\frac{\delta\rho_\sigma}{\rho_\sigma} \sim n\frac{\delta\sigma}{\sigma} + \frac{1}{2}n(n-1)\left(\frac{\delta\sigma}{\sigma}\right)^2 + \frac{1}{6}n(n-1)(n-2)\left(\frac{\delta\sigma}{\sigma}\right)^3. \quad (2.45)$$

Since $\zeta_g \simeq \frac{n+2}{6}f_D\frac{\delta\sigma}{\sigma}$, the curvature perturbation reads

$$\zeta \simeq \zeta_g + \frac{3(n-1)}{(n+2)f_D}\zeta_g^2 + \frac{6(n-1)(n-2)}{(n+2)^2f_D^2}\zeta_g^3. \quad (2.46)$$

Using Eq.(1.1), we find $f_{NL} \simeq \frac{5(n-1)}{(n+2)}\frac{1}{f_D}$ and $g_{NL} \simeq \frac{50(n-1)(n-2)}{3(n+2)^2}\frac{1}{f_D^2}$.

2.2.2 The mass term becomes dominant before curvaton decays

The equation of state of the oscillating curvaton is $p = \frac{n-2}{n+2}\rho$ when $\lambda\sigma^n$ is dominant, and $p = 0$ when the mass term is dominant. In this case, there is a transition from $p = \frac{n-2}{n+2}\rho$ to $p = 0$ when the amplitude of the curvaton oscillations is roughly σ_c . Since the pressure of an oscillating curvaton field is a unique function of its energy density, the energy conservation implies that the curvature perturbation ζ_{σ_o} is conserved [39] even when the equation of state of the oscillating curvaton changes.

In this case, the pressure of the oscillating curvaton is $p = 0$ before it decays and thus

$$\zeta_{\sigma_o} = \zeta + \frac{1}{3} \ln \frac{\rho_{\sigma_o}}{\bar{\rho}_{\sigma_o}}. \quad (2.47)$$

Similarly, on the curvaton-decay hypersurface, we have

$$(1 - \Omega_{\sigma,D})e^{4(\zeta_r - \zeta)} + \Omega_{\sigma,D}e^{3(\zeta_{\sigma_o} - \zeta)} = 1. \quad (2.48)$$

Order by order, the curvature perturbation reads

$$\zeta^{(1)} = f_D\zeta_{\sigma_o}^{(1)}, \quad (2.49)$$

$$\zeta^{(2)} = \left[\frac{3(n+2h_2)}{(n+2)f_D} - 2 - f_D \right] \left(\zeta^{(1)} \right)^2, \quad (2.50)$$

$$\begin{aligned} \zeta^{(3)} = & \left[\frac{9}{(n+2)^2f_D^2} [n(n-2) + 4h_3 + 6nh_2] \right. \\ & \left. - \frac{18}{(n+2)f_D} (n+2h_2) + \frac{2}{n+2} (5 - 2n - 9h_2) + 10f_D + 3f_D^2 \right] \left(\zeta^{(1)} \right)^3, \end{aligned} \quad (2.51)$$

where

$$f_D = \frac{3\Omega_{\sigma,D}}{4 - \Omega_{\sigma,D}}. \quad (2.52)$$

The amplitude of the primordial power spectrum and the non-linearity parameters are

$$P_\zeta = \left(\frac{n+2}{12\pi} \right)^2 f_D^2 q^2 \frac{H_*^2}{\sigma_*^2}, \quad (2.53)$$

$$f_{NL} = \frac{5}{6} \left[\frac{3(n+2h_2)}{(n+2)f_D} - 2 - f_D \right], \quad (2.54)$$

$$\begin{aligned} g_{NL} = & \frac{25}{54} \left[\frac{9}{(n+2)^2f_D^2} [n(n-2) + 4h_3 + 6nh_2] \right. \\ & \left. - \frac{18}{(n+2)f_D} (n+2h_2) + \frac{2}{n+2} (5 - 2n - 9h_2) + 10f_D + 3f_D^2 \right]. \end{aligned} \quad (2.55)$$

For $n = 2$, these results are also the same as those in Sec. 2.1. For $n \neq 2$, if $f_D \ll 1$, $g_{NL} \simeq 25n(n-2)/(6(n+2)^2 f_D^2)$ which is large, and

$$g_{NL} \simeq \frac{2(n-2)}{3n} f_{NL}^2, \quad (2.56)$$

which is different from that in Sec. 2.2.1.

3. Spectral index of the primordial power spectrum and a mixed scenario

The spectral index is an important quantity to characterize the primordial power spectrum. In curvaton model, the scale dependence of the primordial power spectrum is the same as that of $\delta\sigma^{(1)}$. So the spectral index of the primordial power spectrum in the curvaton model takes the form

$$n_s^{cv} = 1 + \frac{2\tilde{m}_*^2}{3H_*^2} - 2\epsilon, \quad (3.1)$$

where \tilde{m}_* is the effective mass of curvaton at $\sigma = \sigma_*$ and $\epsilon = -\frac{\dot{H}_*}{H_*^2}$ is a slow-roll parameter. This result is valid for all of the previous scenarios. Since we have $\tilde{m}_* \ll H_*$ in the curvaton model, a small value of ϵ and a closely scale-invariant power spectrum are expected. However WMAP 5yr data prefers a red-tilted power spectrum. In [26], we suggested a mixed scenario in which the curvature perturbation generated by inflaton also makes a significant contribution to the primordial power spectrum in order to naturally obtain a red-tilted power spectrum in curvaton model. Denote the curvature perturbation generated by curvaton as P_ζ^{cv} . If $P_\zeta^{cv} = \beta P_\zeta^{tot}$, the spectral index of the total primordial power spectrum becomes

$$n_s = \beta n_s^{cv} + (1 - \beta) n_s^{inf}, \quad (3.2)$$

where $n_s^{inf} = 1 - 6\epsilon + 2\eta$ is the spectral index of the power spectrum generated by inflaton. Now the bispectrum and trispectrum are respectively related to the power spectrum by

$$B_\zeta(k_1, k_2, k_3) \simeq \frac{6}{5} f_{NL}^{cv} [\mathcal{P}_\zeta^{cv}(k_1) \mathcal{P}_\zeta^{cv}(k_2) + 2 \text{ perms}], \quad (3.3)$$

$$\begin{aligned} T_\zeta(k_1, k_2, k_3, k_4) &\simeq \tau_{NL}^{cv} [\mathcal{P}_\zeta^{cv}(k_1) \mathcal{P}_\zeta^{cv}(k_2) \mathcal{P}_\zeta^{cv}(k_3) \mathcal{P}_\zeta^{cv}(k_4) + 11 \text{ perms}] \\ &+ \frac{54}{25} g_{NL}^{cv} [\mathcal{P}_\zeta^{cv}(k_1) \mathcal{P}_\zeta^{cv}(k_2) \mathcal{P}_\zeta^{cv}(k_3) \mathcal{P}_\zeta^{cv}(k_4) + 3 \text{ perms}], \end{aligned} \quad (3.4)$$

where we ignore the contribution to the non-linearity parameters from the fluctuation of inflaton. Since $\mathcal{P}_\zeta^{cv} = \beta \mathcal{P}_\zeta^{tot}$, the observed non-Gaussianity parameters become

$$f_{NL} \simeq \beta^2 f_{NL}^{cv}, \quad \tau_{NL} \simeq \beta^3 \tau_{NL}^{cv}, \quad g_{NL} \simeq \beta^3 g_{NL}^{cv}. \quad (3.5)$$

Considering Eq.(2.7), we have

$$\tau_{NL} = \frac{36}{25\beta} (f_{NL})^2. \quad (3.6)$$

In Sec. 2, we conclude that the second order non-Gaussianity parameter g_{NL}^{cv} is proportional to $(f_{NL}^{cv})^2$, i.e. $g_{NL}^{cv} = c(f_{NL}^{cv})^2$ where the coefficient c is different in different case. Similarly we have

$$g_{NL} = \frac{c}{\beta} (f_{NL})^2. \quad (3.7)$$

If $\epsilon \simeq 0$, $n_s \simeq 1 + 2(1 - \beta)\eta$ and a red tilted primordial power spectrum is obtained if $\beta < 1$. Now the second order non-Gaussianity parameters are enhanced by a factor $1/\beta$ for a fixed f_{NL} . Or equivalently, the bound on g_{NL} from experiments will give a bound on β for a given f_{NL} .

4. Curvaton dynamics and the non-linearity parameters

After inflation our universe is dominated by radiation and the Hubble parameter goes like $H = 1/(2t)$. Usually we assume the curvaton field does not evolve until the Hubble parameter drops below the effective mass of curvaton. Once $H \sim \mathcal{O}(\tilde{m})$, the curvaton starts to oscillate around $\sigma = 0$. However the curvaton field slowly evolves even when $H > \tilde{m}$ and the non-linear evolution is also expected if the interaction term is taken into account. The evolution of curvaton after inflation, but before it oscillates, has been discussed in [28, 29, 30, 31, 32] where the interaction term is regarded as a perturbation.

Here we pay attention to the case where the interaction term is dominant before the curvaton starts to oscillate. The curvaton equation of motion after inflation is

$$\ddot{\sigma} + \frac{3}{2t}\dot{\sigma} = -\frac{\lambda}{n-1}\sigma^{n-1}. \quad (4.1)$$

It is difficult to find an analytic solution for this non-linear differential equation. Before curvaton starts to oscillate, the effective curvaton mass is smaller than the Hubble parameter. So the curvaton slowly rolls down its potential. Taking the slow-roll approximation, the curvaton equation of motion is simplified to be

$$\frac{3}{2t}\dot{\sigma} \simeq -\frac{\lambda}{n-1}\sigma^{n-1} \quad (4.2)$$

whose solution with the initial condition $\sigma_{ini} = \sigma_*$ at $t = 0$ is given by

$$\sigma(t) \simeq \sigma_* - \frac{\lambda t^2}{3(n-1)}\sigma_*^{n-1}. \quad (4.3)$$

The curvaton begins to oscillate roughly at the time of $t_o = 1/(2\tilde{m}_o)$ which corresponds to $H = \tilde{m}_o \simeq \sqrt{\lambda\sigma_*^{n-2}}$. Now the parameters q , h_2 and h_3 are respectively given by

$$q \simeq 1 - \frac{x_o^2}{3}, \quad (4.4)$$

$$h_2 \simeq -(n-2)\frac{x_o^2}{3}, \quad (4.5)$$

$$h_3 \simeq -(n-2)(n-3)\frac{x_o^2}{3}, \quad (4.6)$$

where $x_o = \tilde{m}t_o \simeq 1/2$. For $n = 4$, $q \simeq 0.92$, $h_2 \simeq h_3 \simeq -0.17$. The corrections to the non-linearity parameters from the curvaton dynamics after inflation, but before curvaton begins to oscillate, are at the sub-leading order for the case where the interaction term is dominant before it starts to oscillate.

5. Evolution of curvaton after curvaton starts to oscillate

In this section, we assume our universe is dominated by radiation before the curvaton decays in order for a large non-Gaussianity. The case in which the curvaton potential is always dominated by the mass term has been discussed very well. Here we only focus on the evolution of the oscillating curvaton whose potential is dominated by the interaction term when it starts to oscillate.

Assume $a = 1$ at the time of curvaton starting to oscillate. At this time the effective mass of curvaton is

$$\tilde{m}_o = \sqrt{m^2 + \lambda \sigma_*^{n-2}} \simeq \sqrt{\lambda \sigma_*^{n-2}}, \quad (5.1)$$

and the energy density of curvaton is

$$\rho_{\sigma_o} \simeq \frac{\lambda}{n(n-1)} \sigma_*^n. \quad (5.2)$$

Here we assume $\sigma_o = \sigma_* > \sigma_c$. The curvaton energy density drops as $\rho_\sigma = \rho_{\sigma_o} a^{-6n/(n+2)}$. When the universe evolves to

$$a = a_c = \left(\frac{\sigma_*}{\sigma_c} \right)^{\frac{n+2}{6}}, \quad (5.3)$$

the mass term begins to be dominant. The curvaton starts to decay at the time when the Hubble parameter drops below the curvaton decay rate Γ_σ and the scale factor is

$$a_\Gamma = \sqrt{\frac{\tilde{m}_o}{\Gamma_\sigma}}. \quad (5.4)$$

If $a_c < a_\Gamma$, the curvaton potential is dominated by the mass term before it decays, and then

$$\Omega_{\sigma,D} \simeq \frac{\sigma_*^2}{6M_p^2} \frac{m^2}{\tilde{m}_o^{\frac{3}{2}} \Gamma_\sigma^{\frac{1}{2}}} \left(\frac{\sigma_*}{\sigma_c} \right)^{\frac{n}{2}-1}. \quad (5.5)$$

If $a_c > a_\Gamma$, the curvaton potential is always dominated by the interaction term before it decays and we have

$$\Omega_{\sigma,D} \simeq \frac{\sigma_*^2}{3n(n-1)M_p^2} \left(\frac{\tilde{m}_o}{\Gamma_\sigma} \right)^{\frac{4-n}{2+n}}. \quad (5.6)$$

The curvaton energy density does not decrease until it starts to oscillate. So the curvaton energy density increases with respect to radiation before it starts to oscillate for an arbitrary value of n . At the time when the curvaton starts to oscillate ($H = \tilde{m}_o$), we have

$$\Omega_{\sigma,o} = \frac{\lambda \sigma_*^n / (n(n-1))}{3M_p^2 H^2} = \frac{\sigma_*^2}{3n(n-1)M_p^2}, \quad (5.7)$$

which can be $\mathcal{O}(1)$ if $\sigma_* \sim M_p$, even though the energy density of curvaton is negligible during inflation. The energy density of the oscillating curvaton with $n = 4$ goes like a^{-4} which is the same as radiation, and then $\Omega_{\sigma,D} = \Omega_{\sigma,o}$. For $n > 4$, the energy density of the oscillating curvaton decreased with respect to radiation and thus $\Omega_{\sigma,D}$ is suppressed by a

factor $(\Gamma_\sigma/\tilde{m}_o)^{\frac{n-4}{2+n}}$. One point we want to stress is that $\Omega_{\sigma,D}$ can be much larger than the fraction of curvaton energy density in the energy budget during inflation.

Here we also want to estimate the typical value of curvaton during inflation. The behavior of a light scalar field in de Sitter space has been studied in [40, 41, 42, 43]. The quantum fluctuation can be taken as the random walk:

$$\langle \sigma^2 \rangle = \frac{H_*^3}{4\pi^2} t. \quad (5.8)$$

On the other hand, the long wavelength modes of the light scalar field are in the slow-roll regime and obey the slow-roll equation of motion, i.e.

$$3H_* \frac{d\sigma}{dt} = -\frac{dV(\sigma)}{d\sigma} = -m^2\sigma - \frac{\lambda}{n-1}\sigma^{n-1}. \quad (5.9)$$

Combining these two considerations, we have

$$\frac{d\langle \sigma^2 \rangle}{dt} = \frac{H_*^3}{4\pi^2} - \frac{2m^2}{3H_*} \langle \sigma^2 \rangle - \frac{2\lambda}{3(n-1)H_*} \langle \sigma^2 \rangle^{\frac{n}{2}}. \quad (5.10)$$

For $n = 4$, our result is the same as that in [43]. In the case with a dominant interaction term, the solution of the above differential equation approaches a constant equilibrium value

$$\sigma_* \simeq \left(\frac{3(n-1)H_*^4}{8\pi^2\lambda} \right)^{\frac{1}{n}} \quad (5.11)$$

which can be estimated as the typical value of the curvaton during inflation. Now the requirement of $\sigma_* > \sigma_c$ yields

$$H_*^{2n-4} > \frac{3(n-1)}{8\pi^2} \left(\frac{4n\pi^2}{3} \right)^{\frac{n}{2}} \frac{m^n}{\lambda}. \quad (5.12)$$

As a concrete example, we investigate the case of $n = 4$ in the following subsection.

5.1 The case of $n = 4$ with $\sigma_* > \sigma_c$

In this subsection, we estimate the value of curvaton during inflation as the typical value, namely

$$\sigma_* \simeq 0.58\lambda^{-1/4}H_*, \quad (5.13)$$

and the effective mass of curvaton when it starts to oscillate is

$$\tilde{m}_o \simeq 0.58\lambda^{1/4}H_*. \quad (5.14)$$

In this case, $\sigma_c = \sqrt{6}\lambda^{-1/2}m$ and $\sigma_* > \sigma_c$ says

$$m < 0.24\lambda^{1/4}H_*. \quad (5.15)$$

We also have

$$a_c = 0.24\lambda^{1/4}\frac{H_*}{m}, \quad a_\Gamma = 0.76\lambda^{1/8}\sqrt{\frac{H_*}{\Gamma_\sigma}}. \quad (5.16)$$

Requiring $a_c < a_\Gamma$ yields $\Gamma_\sigma \lesssim \Gamma_c = 10\lambda^{-1/4}m^2/H_*$. If the curvaton decay rate is roughly the same as the gravitational strength decay rate $\Gamma_g = m^3/M_p^2$, $a_\Gamma \sim a_c \sqrt{\frac{M_p^2}{mH_*\lambda^{1/4}}} > a_c$ and the mass term becomes dominant before curvaton decays. On the other hand, the gravitational wave perturbation only depends on the inflation scale and H_* is related to the tensor-scalar ratio r by $H_* = 10^{-4}\sqrt{r}M_p$.

- $a_c > a_\Gamma$. The interaction term always dominates the curvaton potential before the curvaton decays. The amplitude of the primordial power spectrum and the non-linearity parameter generated by curvaton field are respectively given by $P_\zeta^{cv} = 0.075\sqrt{\lambda}f_D^2$ and $f_{NL}^{cv} \simeq 5/(2f_D)$. WMAP normalization [11] is $P_{\zeta, wmap} = 2.457 \times 10^{-9}$. If the total amplitude of the primordial power spectrum is contributed by the curvaton fluctuation, we have $P_\zeta^{cv} = P_{\zeta, wmap}$ which implies $\sqrt{\lambda} \geq 3.3 \times 10^{-8}$ because $f_D = \Omega_{\sigma, D} \leq 1$. On the other hand, we have $f_D = \Omega_{\sigma, D} \simeq 9.3 \times 10^{-3}\lambda^{-1/2}H_*^2/M_p^2$, and then

$$f_{NL} = f_{NL}^{cv} = 2.7 \times 10^2 \frac{\sqrt{\lambda}M_p^2}{H_*^2} \gtrsim \frac{891}{r}. \quad (5.17)$$

The limit of r from WMAP5 is $r < 0.2$. The non-Gaussianity parameter f_{NL} is much larger than the upper bound from WMAP 5yr data and the above scenario has been ruled out. On the other hand, in order to make this point clearer, let's start with the bound on f_{NL}^{cv} . Requiring $f_{NL}^{cv} = 5/(2\Omega_{\sigma, D}) = 90M_p^2/\sigma_*^2 \lesssim 111$ yields $\sigma_*^2 \gtrsim 0.81M_p^2$. Since $f_D \leq 1$, $P_\zeta^{cv} = \frac{1}{4\pi^2}f_D^2 \frac{H_*^2}{\sigma_*^2} \lesssim 3 \times 10^{-10}r < P_{\zeta, wmap}$. In order to satisfy the bound on the non-Gaussianity from WMAP 5yr data, it is natural to assume that the fluctuation of inflaton makes a significant contribution to the primordial power spectrum, namely $P_\zeta^{cv} = \beta P_{\zeta, wmap}$. Now the above constraints are released to be $\sqrt{\lambda} \geq 3.3 \times 10^{-8}\beta$ and

$$f_{NL} \geq \frac{891\beta^3}{r}. \quad (5.18)$$

Considering $f_{NL} < 111$ and $r < 0.2$ yields $\beta \lesssim 0.3$ and thus $g_{NL} \gtrsim 3c(f_{NL})^2$. If $f_{NL} = 30$ and $r = 10^{-3}$, $\beta \lesssim 0.03$ and $g_{NL} \gtrsim 3 \times 10^4 c$ which can be detected by Planck.

- $a_c < a_\Gamma$. We also consider $P_\zeta^{cv} = \beta P_\zeta^{tot}$. In this case, $P_\zeta^{cv} = 0.075\sqrt{\lambda}f_D^2$, $f_{NL}^{cv} \simeq 5/(3f_D)$, and then $P_\zeta^{tot} = 0.21\beta^3\sqrt{\lambda}/(f_{NL})^2$. WMAP normalization [11] reads

$$\lambda \simeq 1.4 \times 10^{-16}(f_{NL})^4/\beta^6. \quad (5.19)$$

On the other hand, $f_D \simeq \frac{3}{4}\Omega_{\sigma, D} = 2.3 \times 10^{-2}mH_*^{3/2}/(\lambda^{5/8}M_p^2\Gamma_\sigma^{1/2})$. Considering $f_{NL} = \beta^2 f_{NL}^{cv}$, we have

$$f_{NL} \simeq 23\beta^{7/6}\sqrt{r}\left(\frac{m^2}{M_p\Gamma_\sigma}\right)^{1/3}, \quad (5.20)$$

which is compatible with WMAP 5yr data if the curvaton mass is not too large even when the primordial power spectrum is generated by curvaton field ($\beta = 1$). In this scenario, we have $a_c < a_\Gamma$ and $\sigma_* > \sigma_c$, namely

$$2 \times 10^{-6}r^{3/4}\sqrt{\Gamma_\sigma M_p}/\beta^{1/4} \leq m \leq 2 \times 10^{-22}\frac{r^3}{\beta}\frac{M_p^2}{\Gamma_\sigma}. \quad (5.21)$$

From Eq.(5.20), the lower bound on the curvaton mass is automatically satisfied for $f_{NL} \gtrsim \mathcal{O}(1)$. The upper bound on the curvaton mass leads to a bound on f_{NL} from above as follow

$$f_{NL} \leq 8 \times 10^{-14} \beta^{\frac{1}{2}} r^{\frac{5}{2}} \frac{M_p}{\Gamma_\sigma}. \quad (5.22)$$

A large non-Gaussianity is achieved only when curvaton decay rate is very small compared to M_p . In [30] the author pointed out that the curvaton should decay before neutrino decoupling, namely $\Gamma_\sigma > \Gamma_0 = 1.8 \times 10^{-43} M_p$. Otherwise the curvature perturbations may be accompanied by a significant isocurvature neutrino perturbation. This requirement leads to $f_{NL} \lesssim 4 \times 10^{29} \beta^{\frac{1}{2}} r^{\frac{5}{2}}$ which is quite loose. On the other hand, the curvaton model is free from the constraint of isocurvature perturbation in WMAP 5yr result [11] if the cold dark matter (CDM) is not the direct decay product of the curvaton and CDM is generated after the curvaton decays completely (or equivalently $\Gamma_\sigma \gtrsim (M_{CDM}/20)^2/M_p$) [22]. Combing with Eq.(5.22) yields an upper bound on the mass of CDM

$$M_{CDM} \lesssim 6 \times 10^{-6} \beta^{\frac{1}{4}} r^{\frac{5}{4}} M_p / (f_{NL})^{1/2}. \quad (5.23)$$

In [30] another constraint on Γ_σ is $\Gamma_\sigma \gtrsim \Gamma_g = m^3/M_p^2$ which yields

$$f_{NL} \lesssim 23 \beta^{7/6} \sqrt{r} (M_p/m)^{1/3} \quad (5.24)$$

directly from Eq.(5.20). According to Eq.(5.22), we have

$$f_{NL} \lesssim 8 \times 10^{-14} \beta^{\frac{1}{2}} r^{\frac{5}{2}} \frac{M_p^3}{m^3}, \quad (5.25)$$

which is more restricted than Eq.(5.24) if $\frac{m}{M_p} > 3.8 \times 10^{-6} \beta^{-\frac{1}{4}} r^{\frac{3}{4}}$.

To summarize, if the self-interaction term $\lambda \sigma^4$ is always dominant, $\Omega_{\sigma,D}$ will be too small and the non-Gaussianity is too large to fit the WMAP 5yr data unless the fluctuation of inflaton makes the main contribution to the primordial power spectrum. The constraint on the model where the curvaton potential is always dominated by the interaction term with $n > 4$ before curvaton decays will be more stringent. However, because the energy density of the oscillating curvaton whose potential is dominated by the mass term grows with respect to the radiation, the non-Gaussianity can be compatible with WMAP 5yr data even when the primordial power spectrum is mainly generated by curvaton.

6. Discussions

In this paper we use the $\delta\mathcal{N}$ formalism to calculate the primordial curvature perturbation on large scales in the curvaton model with polynomial potential. The main contribution to the non-Gaussianity in curvatom model comes from the non-linear gravitational perturbations, rather than the curvaton self-interaction. Our calculations are also straightforward to apply to the case with more complicated curvaton potential. When the self-interaction term dominates the curvaton potential during inflation, the order of magnitude of the second order non-linearity parameters τ_{NL} and g_{NL} is roughly $\mathcal{O}(f_{NL}^2)$ if $f_{NL} \gg 1$.

A red-tilted primordial power spectrum can be naturally achieved in curvaton model if the fluctuation of inflaton also makes a significant contribution to it [26]. In this mixed scenario, it is also possible to detect the non-Gaussianity generated during inflation in the generalized inflation models [44, 45, 46, 47, 48, 49, 50, 51, 52]. For another interesting observation in this mixed scenario, the second order non-Gaussianity parameters τ_{NL} and g_{NL} are enhanced for fixed f_{NL} . In addition, multiple curvatons are generically expected in the fundamental theories, such as string theory. It is worth investigating the curvature perturbation in N-vaton [26] with polynomial potential.

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A. Perturbations of a light self-interacting scalar field in the inflationary Universe

In this section we consider the perturbations of a light self-interacting scalar field σ whose energy density is subdominant in the inflationary Universe. The potential of σ is given in Eq.(2.9).

We expand the curvaton field up to the third order in the perturbations around the homogeneous background as

$$\sigma(t, \mathbf{x}) = \sigma(t) + \delta\sigma^{(1)}(t, \mathbf{x}) + \frac{1}{2}\delta\sigma^{(2)}(t, \mathbf{x}) + \frac{1}{6}\delta\sigma^{(3)}(t, \mathbf{x}). \quad (\text{A.1})$$

During inflation the equations of motion for the homogeneous part and the perturbations on large scales in cosmic time are given by

$$\ddot{\sigma} + 3H\dot{\sigma} = -m^2\sigma - \frac{\lambda}{n-1}\sigma^{n-1}, \quad (\text{A.2})$$

$$\delta\ddot{\sigma}^{(1)} + 3H\delta\dot{\sigma}^{(1)} = -\tilde{m}^2\delta\sigma^{(1)}, \quad (\text{A.3})$$

$$\delta\ddot{\sigma}^{(2)} + 3H\delta\dot{\sigma}^{(2)} = -\tilde{m}^2\delta\sigma^{(2)} - (n-2)\lambda\sigma^{n-3}\left(\delta\sigma^{(1)}\right)^2, \quad (\text{A.4})$$

$$\begin{aligned} \delta\ddot{\sigma}^{(3)} + 3H\delta\dot{\sigma}^{(3)} = & -\tilde{m}^2\delta\sigma^{(3)} - 3(n-2)\lambda\sigma^{n-3}\delta\sigma^{(1)}\delta\sigma^{(2)} \\ & - (n-2)(n-3)\lambda\sigma^{n-4}\left(\delta\sigma^{(1)}\right)^3. \end{aligned} \quad (\text{A.5})$$

In slow-roll approximation,

$$3H\delta\dot{\sigma}^{(2)} \simeq -(n-2)\lambda\sigma^{n-3}\left(\delta\sigma^{(1)}\right)^2, \quad (\text{A.6})$$

whose solution is roughly given by [17, 53]

$$\delta\sigma^{(2)} \sim -\frac{N_k}{3H^2}(n-2)\lambda\sigma^{n-3}\left(\delta\sigma^{(1)}\right)^2, \quad (\text{A.7})$$

where $N_k = \int_{t_k}^{t_{end}} H dt$ is the number of e-folds between the end of inflation and the time t_k when the scale of wavenumber k leaves the horizon during inflation. Typically $N_k = 60$. Similarly, the solution of $\delta\sigma^{(3)}$ reads

$$\delta\sigma^{(3)} \sim -\frac{N_k}{3H^2}(n-2)(n-3)\lambda\sigma^{n-4}\left(\delta\sigma^{(1)}\right)^3. \quad (\text{A.8})$$

Both $\delta\sigma^{(2)}$ and $\delta\sigma^{(3)}$ are proportional to the coupling constant. In the limit of $\lambda \rightarrow 0$, we only need to expand the curvaton field to $\delta\sigma^{(1)}$ without higher order, non-Gaussian terms.

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